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Asymptotic Solution of Systems of Linear Ordinary Differential Equations with Discontinuous Coefficients

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DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS
COEFFICIENTS

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Abstract

A system of linear first order ordinary differential equations is considered in which the coefficient matrix depends upon a parameter k. The asymptotic form of the solutions of this system are determined for large values of k when the coefficient matrix or some of its derivatives have finite jump discontinuities. The asymptotic form obtained contains terms beyond those previously known. These new terms have discontinuous coefficients, whereas the previously known terms are continuous.

The results are applied to the one dimensional propagation of time-harmonic waves. It is shown that in this case the reflection coefficient is proportional to k^{-J} if the J-th derivative of the index of refraction is discontinuous and the lower order derivates are continuous. Here $k = 2\pi/\lambda$ is the propagation constant and λ the wavelength in a medium of unit refractive index. This result is to be contrasted with the fact that the reflection coefficient vanishes faster than any negative power of k if all derivatives of the index of refraction are continuous.

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1.   Introduction

Let us consider the vector solution $u(x,k)$ of the system of N first order linear ordinary differential equations

$$(1) \qquad \frac{du}{dx} = kA(x,k)u$$

In this system the coefficient matrix kA depends upon a real parameter k. Asymptotic expansions of u in powers of k^{-1}, which are valid as $k \to \infty$, are well known. They hold provided that A has continuous derivatives of all orders and satisfies certain other conditions. But if some derivative of A is discontinuous, then only a finite number of terms in the expansion of u are known. We consider the problem of finding additional terms when A and its derivatives have finite jump discontinuities. We shall show how to solve this problem, and we shall find explicitly the next term beyond those previously known. This term is discontinuous, whereas the preceding terms are all continuous. This fact is significant in various applications.

One important application of our result is to one dimensional propagation of time harmonic waves in an inhomogeneous medium. Such a wave motion w satisfies the reduced wave equation

$$(2) \qquad \frac{d^2w}{dx^2} + k^2n^2(x)w = 0.$$

Here $n(x)$ denotes the refractive index of the medium and k denotes the propagation constant in a medium with $n = 1$. It follows from known results that the reflection coefficient $R(k)$ determined from (2) is asymptotically zero to all orders in k^{-1}, provided that n has continuous derivatives of all orders. However, if the J-th derivative of n has a finite jump discontinuity, and the

lower order derivatives are continuous, our result shows that $R(k)$ is of the order k^{-J}. This essential difference between the infinitely and finitely differentiable cases has been observed by S.A. Schelkunoff[1] and J. Feinstein[2]. Equation (2) is treated in section 4.

2. Preliminaries

We wish to determine the asymptotic behavior, for large values of k, of the solutions of (1) in an interval I of the real x axis. We assume that within I the matrix A has the asymptotic expansion

$$(3) \qquad A(x,k) \sim \sum_{j=o}^{\infty} A_j(x)k^{-j}.$$

We also assume that $A_o(x)$ has N eigenvalues $\lambda_1(x),\ldots\lambda_N(x)$ which are distinct in I. Then there exists a non singular matrix $R(x)$ which diagonalizes $A_o(x)$ in I. We define the diagonal matrix $\Lambda(x)$ by the equation

$$(4) \qquad \Lambda(x) = R^{-1}A_o R \equiv \mathrm{diag}\left[\lambda_1(x),\ldots,\lambda_N(x)\right].$$

The matrix R can be so chosen that, like Λ, it is independent of k and possesses as many continuous derivatives as does $A_o(x)$.

With the aid of R we define the vector $v(x,k)$ by the equation

$$(5) \qquad u(x,k) = R(x)v(x,k).$$

Then from (1), it follows that v satisfies the equation

$$(6) \qquad \frac{dv}{dx} = kB(x,k)v.$$

Here the matrix B is defined by

$$(7) \qquad B(x,k) = R^{-1}AR - k^{-1}R^{-1}dR/dx.$$

It has the asymptotic expansion

$$(8) \qquad B(x,k) \sim \sum_{j=0}^{\infty} B_j(x)k^{-j}.$$

The B_j are given by

$$(9) \qquad B_o = \Lambda, \quad B_1 = R^{-1}A_1R - R^{-1}dR/dx, \quad B_j = R^{-1}A_jR, \quad j = 2,3,\dots .$$

We shall determine the asymptotic behavior of the solutions v of (6). Then the behavior of the solutions u of (1) will be found by means of (5). The advantage of dealing with (6) rather than (1) is that B_o is diagonal while A_o is not necessarily diagonal.

Let us tentatively assume that (6) has N linearly independent solutions $v_i(x,k)$ with asymptotic expansions of the form

$$(10) \qquad v_i(x,k) \sim \exp\left[k \int_{x_o}^{x} \lambda_i(x)dx\right]\left(\sum_{s=0}^{\infty} v_i^s(x)k^{-s}\right) \quad i = 1,\dots,N.$$

Let us also suppose that the expansion of dv/dx is obtained by termwise differentiation of (10). Then we can insert (10) and (8) into (6) and equate to zero the coefficient of each power of k^{-1}. In this way we obtain the following set of equations:

(11) $\qquad M_i v_i^s = dv_i^{s-1}/dx - \displaystyle\sum_{j=0}^{s-1} B_{s-j} v_i^j \qquad s = 0,1,\ldots$.

Quantities with negative subscripts or superscripts in (11) and elsewhere are defined to be zero, and M_i is defined by

(12) $\qquad M_i = B_0 - \lambda_i I = \mathrm{diag}\left[\lambda_1 - \lambda_i,\ldots,\lambda_N - \lambda_i\right]$.

M_i is diagonal with a zero in the (i,i) position and it is therefore a singular matrix.

If we set $s = 0$ in (11) we see that the right side vanishes. Therefore the only non zero component of v_i^0 is the ith one. Then the solvability condition for (11) with $s \neq 0$ is simply that the ith component of the right side vanish. Upon replacing s by $s + 1$, this condition can be written in the form

(13) $\qquad dv_{ii}^s/dx - (B_1 v_i^s)_i = \displaystyle\sum_{j=0}^{s-1} (B_{s+1-j} v_i^j)_i \qquad s = 0,1,\ldots$.

The second subscript indicates the component of the vector. Equation (13) is a first order ordinary differential equation for v_{ii}^s. From (11) all other components of v_i^s can be determined in terms of the v_i^j with $j < s$. Then v_{ii}^s can be found from (13). Thus by starting with $s = 0$, we can determine successively all the v_i^s.

By keeping track of the differentiation which is performed in (11) at each step of this process, we can determine the continuity properties of the derivatives of the v_i^s in terms of those of the B_j and therefore of the A_j. In this way we arrive at the following lemma, which is due to P.D. Tamarkin [3].

Lemma   If for some $J_o \geq 0$ and all $j \leq J_o$, each coef-
ficient A_j has $J_o - j$ continuous derivatives, then
so does v_i^j for $i = 1,\ldots,N$.

This lemma can be verified readily by induction, with the aid of (11) and (13).
Tamarkin has also proved the following theorem.

Theorem I   Suppose that in an interval I of the x axis
$A(x,k)$ has the asymptotic form

$$(14) \qquad A(x,k) = \sum_{j=0}^{J_o} A_j(x)k^{-j} + o(k^{-J_o}).$$

Let the A_j satisfy the hypothesis of the lemma and let
$A_o(x)$ have distinct eigenvalues $\lambda_i(x)$, $i = 1,\ldots,N$ in
I. Let $u_i^s(x) = Rv_i^s$ where v_i^s are the vectors constructed
as described above and R is a matrix which diagonalizes
A_o and has as many continuous derivatives as does A_o.
Then there exist N linearly independent solutions of
(1), $u_i(x,k)$, $i = 1,\ldots,N$, such that in I

$$(15) \qquad u_i(x,k) = \exp\left[k \int_{x_o}^{x} \lambda_i(x)dx\right]\left(\sum_{j=0}^{J_o} u_i^s(x)k^{-s} + o(k^{-J_o})\right)$$

The coefficient $u_i^s(x)$ has $J - s$ continuous derivatives.

3.   The discontinuous case

Let us now suppose that J is the largest value of J_o for which the hypo-
thesis of the lemma is satisfied. Then for at least one value of j in the
range $0 \leq j \leq J + 1$, the coefficient A_j has a discontinuous $(J + 1 - j)$th

derivative. Let us suppose that this discontinuity occurs at x_0. Then we may choose I so small that for each j in the range $0 \leqq j \leqq J + 1$, the $(J + 1 - j)$-th derivative of A_j is continuous in I, except perhaps at x_0. We further assume that the only discontinuities of any of these derivatives are finite jump discontinuities at x_0. From the definition (9) of the B_j if follows that the properties just described pertain to them also.

Let us now consider the solutions $v(x,k)$ of (6) in which the coefficient matrix is $B(x,k)$. With the asumption just made on the B_j, the hypotheses of the lemma and of Theorem I are satisfied by B with $J_0 = J + 1$, in each of the two subintervals I^- and I^+ into which x_0 divides I. Therefore there are N linearly independent solutions v_i of (6) which have the asymptotic form (15) in I^- with $J_0 = J + 1$, and N others w_i which have that form in I^+. We wish to determine the asymptotic form of, say, the v_i in I^+. To do this we first choose w_i so that its asymptotic form coincides with that of v_i throughout I, up to and including terms of order k^{-J}. That this is possible follows from Theorem I since we may choose the coefficients $w_i^j = v_i^j$ for $j = 0,\ldots,J$ and then determine w_i^{J+1} in I^+ by the construction of the previous section. Since the component w_{ii}^{J+1} is obtained by integrating a first order differential equation, its value can be arbitrarily prescribed at one point. Therefore for later convenience we prescribe

(16) $w_{ii}^{J+1}(x_0) = v_{ii}^{J+1}(x_0).$

Then the theorem guarantees the existence of a solution w_i with these coefficients.

By the linear independence of the w_i we may write

(17) $v_i(x,k) = \displaystyle\sum_{s=0}^{N} c_{is}(k)w_s(x,k).$

Since the coefficients in the asymptotic form of w_i coincide with those of v_i through terms in k^{-J} it follows that the constant c_{is} differs from the Kronecker δ_{is} by terms of order k^{-J-1}. Therefore there are constants γ_{is} such that

(18) $\qquad c_{is}(k) = \delta_{is} + \gamma_{is} k^{-J-1} + o(k^{-J-1}).$

When (18) and the asymptotic form of w_s for x in I^+ are inserted into (17), the following asymptotic form of v_i is obtained in I^+:

(19) $\qquad v_i(x,k) = \exp\left[k \int_{x_0}^{x} \lambda_i(x) dx\right] \sum_{s=0}^{J+1} w_i^s(x) k^{-s} + \sum_{t=1}^{N} \exp\left[k \int_{x_0}^{x} \lambda_t(x) dx\right]$

$$\times \gamma_{it} w_t^0(x) \left[k^{-J-1} + o(k^{-J-1})\right], \qquad x \geq x_0.$$

Upon comparing (19) with (15) we see that whereas the asymptotic form of v_i contains only one of the exponentials in I^-, it contains all of them in I^+. To determine c_{is} we set $x = x_0$ in (17) and solve the resulting linear equations. They have a unique solution because the w_s are linearly independent. In this way c_{is} is given in terms of the $v_i(x_0,k)$ and $w_i(x_0,k)$. Since x_0 belongs to both I^- and I^+, both v_i and w_i have asymptotic forms of the type (15) at x_0, with $J_0 = J + 1$. By inserting them into the expression for c_{is} the asymptotic form of c_{is} is determined. In particular, to determine γ_{it} we set $x = x_0$ in (19) and use the asymptotic form of v_i given by (15), which applies at $x = x_0$. Then the coefficients of k^{-J-1} in (19) yield

(20) $\qquad v_i^{J+1}(x_0) = w_i^{J+1}(x_0) + \sum_{t=1}^{N} \gamma_{it} w_t^0(x_0), \qquad i = 1,\ldots,N.$

To solve (20) we recall that the only nonzero component of w_t^o is w_{tt}^o because w_t^o is an eigenvector of the diagonal matrix B_o. Therefore only one γ_{it} occurs in each component of the right member of (20). Solving (20) yields

$$(21) \qquad \gamma_{it} = \frac{v_{it}^{J+1}(x_o) - w_{it}^{J+1}(x_o)}{w_{tt}^o(x_o)} \; .$$

From (16) and (21) we see that

$$(22) \qquad \gamma_{ii} = 0 .$$

We shall now compute the right member of (21) in terms of the coefficients B_j, and thus express the γ_{it} in terms of them. For this purpose it is convenient to introduce the special jump notation

$$(23) \qquad \left[v_i^j\right] = w_i^j(x_o+) - v_i^j(x_o-) \; .$$

When applied to B_j or M_j, the jump notation will have its usual meaning

$$(24) \qquad \left[B_j\right] = B_j(x_o+) - B_j(x_o-) \; .$$

Let us now consider (11) with $s = J + 1$. We write (11) for v_i^{J+1} at x_o- and for w_i^{J+1} at x_o+ and subtract the two equations. By hypothesis M_i and B_{J+1-j} with $j = 1,\ldots,J$ are continuous. By construction $v_i^j(x_o) = w_i^j(x_o)$ for $j = 0,\ldots,J$. Thus we obtain

$$(25) \qquad M_j\left[v_i^{J+1}\right] = \left[dv_i^J/dx\right] - \left[B_{J+1}\right]v_i^o \; .$$

The left side of (25) contains the jumps we need in (21) to compute γ_{it}. To determine them we must eliminate the derivative from (25). To this end we now write (11) with $s = J$ and differentiate it with respect to x obtaining

$$(26) \qquad \frac{dM_i}{dv} v_i^J + M_i \frac{dv_i^J}{dx} = \frac{d^2 v_i^{J-1}}{dx^2} - \sum_{j=0}^{J-1} \frac{dB_{J-j}}{dx} v_i^j + B_{J-j} \frac{dv_i^j}{dx} \qquad .$$

By construction $v_i^j(x_o) = w_i^j(x_o)$ for $j = 0,\ldots,J$ and by hypothesis dM_i/dx is continuous if $J > 0$. Also by hypothesis dB_{J-j}/dx is continuous for $j = 1,\ldots,J-1$ and B_{J-j} for $j = 0,\ldots,J-1$. By the lemma dv_i^j/dx is continuous for $j = 0,\ldots,J-1$. Thus upon subtracting (26) at x_o- from the corresponding equation for w at x_o+, we obtain

$$(27) \qquad M_i \left[\frac{dv_i^j}{dx} \right] = \left[\frac{d^2 v_i^{J-1}}{dx^2} \right] - \left[\frac{dB_J}{dx} \right] v_i^o .$$

We now multiply (25) by M_i and combine it with (27) to yield

$$(28) \qquad (M_i)^2 \left[v_i^{J+1} \right] = \left[\frac{d^2 v_i^{J-1}}{dx^2} \right] - \left(\left[\frac{dB_J}{dx} \right] + M_i \left[B_{J+1} \right] \right) v_i^o .$$

By continuing in the same way we obtain for any $m \leq J + 1$

$$(29) \qquad (M_i)^m \left[v_i^{J+1} \right] = \left[\frac{d^m v_i^{J-m+1}}{dx^m} \right] - \sum_{r=0}^{m-1} (M_i)^r \left[\frac{d^{m-r-1} B_{J-m+r+2}}{dx^{m-r-1}} \right] v_i^o .$$

Specializing (29) to $m = J + 1$ yields

$$(30) \qquad (M_i)^{J+1} \left[v_i^{J+1} \right] = \left[\frac{d^{J+1} v_i^o}{dx^{J+1}} \right] - \sum_{r=0}^{J} (M_i)^r \left[\frac{d^{J-r} B_{r+1}}{dx^{J-r}} \right] v_i^o .$$

To eliminate the derivative of v_i^o we multiply (30) by M_i. Since the only non-zero element of v_i^o is v_{11}^o, and since the ii element of M_i is zero, the product $M_i \left[d^{J+1} v_i^o / dx^{J+1} \right] = 0$. Then (30) becomes

$$(31) \qquad (M_i)^{J+2}\left[v_i^{J+1}\right] = -\sum_{r=0}^{J} (M_i)^{r+1} \left[\frac{d^{J-r}B_{r+1}}{dx^{J-r}}\right] v_i^o , \qquad x = x_o.$$

Since M_i is diagonal with a zero in the ii position (31) can be solved for the components $\left[v_{it}^{J+1}\right]$ with $t \neq i$. The result is

$$(32) \qquad \left[v_{it}^{J+1}\right] = -\sum_{r=0}^{J} (\lambda_t - \lambda_i)^{r-J-1} \left[\frac{d^{J-r}B_{r+1,ti}}{dx^{J-r}}\right] v_{ii}^o , \qquad i \neq t .$$

Finally, (32) and (21) yield

$$(33) \qquad \gamma_{it} = -\frac{v_{ii}^o}{v_{tt}^o} \sum_{r=0}^{J} (\lambda_t - \lambda_i)^{r-J-1} \left[\frac{d^{J-r}B_{r+1,ti}}{dx^{J-r}}\right] , \qquad x = x_o , i \neq t .$$

This result together with (22) completely determine γ_{it}. Thus the asymptotic form (19) of v_i in I^+ is determined. Then the asymptotic form of u_i, the solution of (1), is given by multiplying the asymptotic form of v_i by the matrix R, according to (5).

We may summarize our results as

> Theorem II Suppose that in an interval I of the x axis,
> $A(x,k)$ has the asymptotic form
>
> $$(34) \qquad A(x,k) = \sum_{j=0}^{J+1} A_j(x)k^{-j} + o(k^{-J-1}), \qquad J \geqq 0 .$$
>
> Let $A_o(x)$ have N distinct eigenvalues $\lambda_i(x)$, i=1,...,N,
> in I. Furthermore, let A_j have J-j continuous deriva-
> tives in I for j=0,...,J. Suppose that the only discon-

tinuities in I of the derivatives of order $(J + 1 - j)$,
$j = 0,\ldots,J+1$ are finite jumps at x_o. Let $u_i^s(x)=R(x)v_i^s(x)$
and $\tilde{u}_i^s(x)=Rw_i^s(x)$, $i = 1,\ldots,N$ and $s = 0,\ldots,J+1$, where
v_i^s and w_i^s are the vectors constructed as described above
and R is a matrix which diagonalizes A_o and has as many
continuous derivatives as does A_o. Let the constants γ_{it}
be given by (33) for $i \neq t$, and $\gamma_{ii} = 0$. The coefficients
u_i^s and $\tilde{u}_i^s$ have J-s continuous derivatives and at x_o these
derivatives of u_i^s equal those of $\tilde{u}_i^s$ for $s = 0,\ldots,J$. Then
there exist N linearly independent solutions of (1),
$u_i(x,k)$, $i = 1,\ldots,N$ with the asymptotic forms

$$(35) \qquad u_i(x,k) = \exp\left[k \int_{x_o}^{x} \lambda_i(x)dx\right] \left[\sum_{s=o}^{J+1} u_i^s(x)k^{-s}+ o(k^{-J-1})\right] \qquad x \leq x_o$$

$$(36) \qquad u_i(x,k) = \exp\left[k \int_{x_o}^{x} \lambda_i(x)dx\right] \left[\sum_{s=o}^{J+1} \tilde{u}_i^s(x)k^{-s}+ o(k^{-J-1})\right]$$

$$+ \sum_{t=1}^{N} \exp\left[k \int_{x_o}^{x} \lambda_t(x)dx\right] \left[\gamma_{it}\tilde{u}_t^o(x)k^{-J-1}+ o(k^{-J-1})\right] \qquad x \geq x_o$$

4.   Application

Let us now consider the asymptotic form of the solutions of (2) when there
is a finite jump in the (J+1)-st derivative of the coefficient $n^2(x)$. This
coefficient is the square of the refractive index in wave propagation. We first
introduce the two component vector u defined by

$$(37) \qquad u = \begin{pmatrix} kw \\ w' \end{pmatrix}$$

Then it follows from (2) that u satisfies (1) with A given by

$$(38) \qquad A(x) = \begin{pmatrix} 0 & 1 \\ -n^2(x) & 0 \end{pmatrix}$$

We see that A is of the form (3) with $A_0 = A$, $A_j \equiv 0$ for $j \geq 1$. The eigenvalues of $A_0(x)$ are $\lambda_1 = in(x)$ and $\lambda_2 = -in(x)$, which are distinct if $n \neq 0$. If the first J derivatives of n are continuous, while the (J+1)-st derivative has a jump at x_0, the same is true of A_0 provided $n(x_0) \neq 0$. Thus the hypotheses of Theorem II are satisfied and therefore (1) has two linearly independent solutions with the asymptotic forms (35) and (36).

If $n(x) > 0$ we may interpret the factors

$$\exp\left[ik \int_{x_0}^{x} ndx\right] \text{ and } \exp\left[-ik \int_{x_0}^{x} ndx\right]$$

as representing waves traveling to the right and left respectively. Then the solution $u_2(x)$ represents a wave traveling to the left in the region $x \leq x_0$, according to (35). However, by (36) it represents both left and right traveling waves in the region $x \geq x_0$. We may interpret this solution as corresponding to an incident wave traveling to the left in the region $x > x_0$ and being partly transmitted and partly reflected at $x = x_0$. The reflected wave is of the order k^{-J-1} with respect to the incident wave, as we see from (36). Equation (36) also shows that the first component of the reflected wave, which is kw according to (37), is proportional to γ_{21}. Thus the reflection coefficient of the discontinuity for incidence from the right is proportional to $\gamma_{21}k^{-J-1}$.

To determine γ_{21} we could use (33). However, we shall instead analyze (2) directly, without transforming it into a first order system because the direct analysis is usually simpler for a single equation. Therefore, our present calculations will be independent of the preceding ones. We shall seek two linearly

independent solutions $u_1(x,k)$ and $u_2(x,k) = \bar{u}_1(x,k)$, the complex conjugate of u_1. Our analysis will show that u_2 has the asymptotic form

$$(39) \qquad u_2(x,k) = \exp\left[-ik \int_{x_0}^{x} n\,dx\right]\left[\sum_{j=0}^{J+1} v_j(x)(-ik)^{-j} + o(k^{-J-1})\right] \qquad x \le x_0$$

$$(40) \qquad u_2(x,k) = \exp\left[-ik \int_{x_0}^{x} n\,dx\right]\left[\sum_{j=0}^{J+1} v_j(x)(-ik)^{-j} + o(k^{-J-1})\right]$$

$$+ \exp\left[ik \int_{x_0}^{x} n\,dx\right]\left[\gamma v_0(x)(-ik)^{-J-1} + o(k^{-J-1})\right] \qquad x \ge x_0$$

Upon inserting (39) or (40) into (2) and equating to zero the coefficient of each power of k, we obtain

$$(41) \qquad 2n\, v_j' + n' v_j = -v_{j-1}'' \qquad\qquad j = 0,\ldots,J+1$$

The continuity of u_2 at x_0 yields

$$(42) \qquad \left[v_j\right] = 0 \qquad\qquad j = 0,\ldots,J$$

$$(43) \qquad \gamma = -\left[v_{J+1}\right]/v_0(x_0)$$

Continuity of u_2' at x_0 yields

$$(44) \qquad \left[v_j\right] = -\left[v_{j-1}'\right]/n(x_0) \qquad\qquad j = 0,\ldots,J$$

$$(45) \qquad n(x_0)\left[v_{J+1}\right] = \gamma n(x_0) v_0(x_0) - \left[v_J'\right]$$

From (42) and (44) we find

$$(46) \qquad \left[v_j'\right] = 0 \qquad\qquad j = 0,\ldots,J-1$$

From (43) and (45) we obtain

$$(47) \qquad \gamma = \left[v_J'\right]/2v_0(x_0)n(x_0)$$

To calculate γ we first note that we can find solutions of (41) such that v_j has $J - j$ continuous derivatives, for $j = 0,\ldots,J$. Then (42) and (46) are satisfied. Then we differentiate (41) m times and subtract the result at x_0- from that at x_0+ obtaining

$$(48) \qquad 2n(x_0)\left[v_j^{(m)}\right] + v_j(x_0)\left[n^{(m)}\right] = -\left[v_{j-1}^{(m+1)}\right], \quad m=1,\ldots,J-j+1; j=0,\ldots,J$$

Upon setting $m = J-j+1$ in (48), we obtain for $j \neq 0$,

$$(49) \qquad \left[v_j^{(J-j+1)}\right] = -\left[v_{j-1}^{(J-j+2)}\right]/2n(x_0), \quad j=1,\ldots,J$$

If we set $j = 0$ and $m = J+1$ in (48) we get

$$(50) \qquad \left[v_0^{(J+1)}\right] = -\left[n^{(J+1)}\right]v_0(x_0)/2n(x_0)$$

Now we set $j = J+1$ in (49), and then using (49) repeatedly, we obtain

$$(51) \qquad \left[v_{J+1}\right] = -\left[v_J'\right]/2n(x_0) = \left(\frac{-1}{2n(x_0)}\right)^2\left[v_{J-1}^{(2)}\right]$$

$$= \left(\frac{-1}{2n(x_0)}\right)^{J+2}v_0(x_0)\left[n^{(J+1)}\right]$$

Thus (43) or (47) yields

$$(52) \qquad \gamma = -\left(\frac{-1}{2n(x_0)}\right)^{J+2}\left[n^{(J+1)}\right]$$

On the basis of these results, we can determine the reflection coefficient of the variable medium with index $n(x)$. For this purpose we must suppose that $n(x)$ tends to the value one as x tends to $\pm \infty$. In fact, $n(x)$ must tend to one fast enough so that $n(x) - 1$ is integrable. We now observe that u_2 represents a wave incident from $+ \infty$ which is partly transmitted and partly reflected at

$x = x_o$. The amplitudes A_{inc} and A_{ref} of the incident and reflected waves are defined as

$$(53) \qquad A_{inc} = \exp\left[-ik\left(\int_{x_o}^{\infty} (n-1)dx - x_o\right)\right] v_o(\infty) \ ,$$

$$(54) \qquad A_{ref} = \exp\left[ik\left(\int_{x_o}^{\infty} (n-1)dx - x_o\right)\right] \gamma k^{-J-1} v_o(\infty) \ .$$

Then the reflection coefficient R_1 is given by

$$(55) \qquad R_1 = \frac{A_{ref}}{A_{inc}} = \gamma(-ik)^{-J-1} \exp\left[2ik\left(\int_{x_o}^{\infty} (n-1)dx - x_o\right)\right]$$

$$= \frac{1}{2n} \left(\frac{1}{2ikn(x_o)}\right)^{J+1} \left[\frac{d^{J+1}n}{dx^{J+1}}\right] \exp\left[2ik\left(\int_{x_o}^{\infty} (n-1)dx - x_o\right)\right] \ .$$

To determine the reflection coefficient for waves incident from the left, we must construct a solution which contains only a transmitted wave for $x > x_o$. Such a solution is $u_1 - \gamma(-ik)^{-J-1} u_2$, at least through terms of order k^{-J-1}. From it we find

$$(56) \qquad R_2 = -\gamma(-ik)^{-J-1} \exp\left[-2ik\left(\int_{-\infty}^{x_o} (n-1)dx + x_o\right)\right]$$

$$= -\frac{1}{2n(x_o)} \left(\frac{1}{2ikn(x_o)}\right)^{J+1} \left[\frac{d^{J+1}n}{dx^{J+1}}\right] \exp\left[-2ik\left(\int_{-\infty}^{x_o} (n-1)dx + x_o\right)\right] \ .$$

5.   The case of a discontinuous A_o

Let us finally consider the case in which A_o is discontinuous, which was not included in the preceding sections. In this case the matrix R is not necessarily continuous, and therefore we do not introduce it. Instead we work directly with (1) and prove the following

Theorem III Suppose that in an interval I of the x axis, $A(x,k)$ has for large k the asymptotic form

(57) $A(x,k) = A_o(x) + o(1)$.

Let $A_o(x)$ have N distinct eigenvalues $\lambda_i(x)$, $i = 1,\ldots,N$, in I and let A_o be continuous in I except for a finite jump at x_o. Then there are N linearly independent solutions of (1), $u_i(x,k)$, $i = 1,\ldots,N$, with the following asymptotic forms for large k

(58) $u_i(x,k) = \exp\left[k \int_{x_o}^{x} \lambda_i(x)dx\right]\left[u_i^o(x) + o(1)\right], \qquad x \leqq x_o$

(59) $u_i(x,k) = \sum_{t=1}^{N} \exp\left[k \int_{x_o}^{x} \lambda_t(x)dx\right]\left[c_{it}^o \tilde{u}_i^o(x) + o(1)\right], \qquad x \geqq x_o$

To prove this result we first apply theorem I with $J_o = 0$ in the region $x \leqq x_o$ to establish the existence of N linearly independent solutions u_i for which (58) holds. Then we apply it again for $x \geqq x_o$ to obtain N other linearly independent solutions $\tilde{u}_i$ which have there asymptotic forms of the type shown in (58) with coefficients $\tilde{u}_i^o(x)$ in place of $u_i^o(x)$. Next we express u_i in terms of the $\tilde{u}_i$ in the form

$$(60) \qquad u_i(x,k) = \sum_{t=1}^{N} c_{it}(k)\tilde{u}_t(x,k).$$

We then apply (60) at $x = x_o$ using the asymptotic form (58) for u_i and the corresponding one for $\tilde{u}_t$. The resulting equation shows that $c_{it} = c_{it}^o + o(1)$ where c_{it}^o is determined by the equations.

$$(61) \qquad u_i^o(x_o) = \sum_{t=1}^{N} c_{it}^o \tilde{u}_i^o(x_o), \qquad i = 1,\ldots,N.$$

The solvability of (61) is assured by the linear independence of the $\tilde{u}_i(x)$. When the asymptotic form of c_{it} and that of $\tilde{u}_t(x)$ in the region $x \le x_o$ are inserted into (60), (59) results. This completes the proof of the theorem, which is essentially the extension of Theorem II to the previously excluded case $J = -1$. The results of (52), (55) and (56) of Section 4 also hold for $J = -1$ if $2n(x_o)$ in them is replaced by $n(x_o+) + n(x_o-)$.

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